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Matrix model for discretized moduli space

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We study the algebraic geometrical background of the Penner–Kontsevich matrix model with the potential $N\alpha\ \mathrm{tr}[-\frac{1}{2}\Lambda X\Lambda X+\log(1-X)+X]$. We show that this model describes intersection indices of linear bundles on the discretized moduli space just in the same fashion as the Kontsevich model is related to intersection indices (cohomological classes) on Riemann surfaces of arbitrary genus. The special role of the logarithmic potential originating from the Penner matrix model is demonstrated. The boundary effects, which were not essential in the case of the Kontsevich model, are now relevant, and the intersection indices on the discretized moduli space of genus g are expressed through Kontsevich's indices of genus g and of the lower genera using a stratification procedure.

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1. Introduction

The last three years of development in matrix models initiated by refs. [1] have revealed many applications of these models in various branches of mathematical physics: two-dimensional quantum field theory, intersection theory on the moduli space of Riemann surfaces, integrable hierarchies, matrix integrals, random surfaces and others. The approach of refs. [1], where the explicit solution was presented, deals with triangulated Riemann surfaces where any triangulation determines some singular metric obtained by the arrangement of equilateral triangles. One may think that when the number of triangles tends to infinity these singular metrics approximate "random metrics" on the surface. These triangulations were presented by a hermitean $N\times N$ one-matrix model

$$\int \exp(\mathrm{tr}\, P(X))\,\mathrm{DX}\,, \qquad (1.1)$$

where $P(X)=\sum_n T_n\,\mathrm{tr}\,X^n$, T_n being times for the one-matrix model. For this system discrete Toda chain equations hold with an additional Virasoro symmetry

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imposed [2]. In the limit $N\to\infty$ the Korteweg–de Vries equation arises. The partition function of two-dimensional gravity for this approach is a series in an infinite number of variables and coincides with the logarithm of some τ-function for the KdV hierarchy.

Another approach to two-dimensional gravity is to evaluate the integral over all classes of conformally equivalent metrics on Riemann surfaces. It may be represented as an integral over the finite-dimensional space of conformal structures. This integral has a cohomological description as an intersection theory on the compactified moduli space of complex curves. Edward Witten presented compelling evidence for a relationship between random surfaces and the algebraic topology of moduli space [3,4]. In fact, he suggested that these expressions coincide since both satisfy the same equations of the KdV hierarchy. It was Maxim Kontsevich who proved this assumption [5]. Surprisingly, he explicitly presented a new matrix model defining exactly the values of the intersection indices or, in the language of 2D gravity, correlation functions of observables $\mathcal{O}_n$ of the type

$$\langle \mathcal{O}_{n_1}\cdots\mathcal{O}_{n_s}\rangle_g , \tag{1.2}$$

where $\langle\cdots\rangle_g$ denotes the expectation value on a Riemann surface with g handles. Then the string partition function $\tau(t)$ has an asymptotic expansion of the form

$$\tau(t)=\exp\sum_{g=0}^{\infty}\left\langle\exp\sum_n t_n\mathcal{O}_n\right\rangle_g , \tag{1.3}$$

and it is a τ-function of the KdV hierarchy taken at a point of the Grassmannian where it is invariant under the action of the set of Virasoro constraints: $\mathscr{L}_n\tau(t)=0$, $n\geqslant -1$ [6–9]. One might say that the Kontsevich model is used to triangulate moduli space, whereas the original models triangulated Riemann surfaces (see, e.g., ref. [10]).

The generalization of the Kontsevich model, the so-called Generalized Kontsevich Model (GKM) [11], is related to the two-dimensional Toda lattice hierarchy and it originated from the external field problem defined by the integral

$$Z[\Lambda;N]=\int DX\exp\{N\,\mathrm{tr}(\Lambda X-V_0(X))\} , \tag{1.4}$$

where $V_0(X)=\sum_n t_n\,\mathrm{tr}\,X^n$ is some potential and t_n are related to times of the hierarchy. This model is equivalent to the Kontsevich integral for $V_0(X)\sim\mathrm{tr}\,X^3$. To solve the integral (1.4) one may use the Schwinger–Dyson equation technique [12] written in terms of eigenvalues of Λ. The Kontsevich model was solved in the genus expansion in refs. [8,13] for genus zero (planar diagrams) and in ref. [14] for higher genus.

Recently, the Kontsevich–Penner model was introduced [15]. The Lagrangian of this model has the following form:

$$\mathscr{Z}[\Lambda]=\int DX\exp(N\,\mathrm{tr}\{-\tfrac{1}{2}\Lambda X\Lambda X+\alpha[\log(1+X)-X]\}) ,$$

$$\Lambda=\mathrm{diag}(\Lambda_1,...,\Lambda_N) . \qquad (1.5)$$

This model may be readily reduced to (1.4) with $V_0(X)=-X^2/2+\alpha\log X$. It was solved in the genus expansion in refs. [15,16]. It was shown in refs. [17,18] that it is in fact equivalent to the one-matrix hermitean model (1.1) with the general potential

$$P(X)=\sum_{n=0}^{\infty}T_n\,\mathrm{tr}\,X^n , \qquad (1.6)$$

of which the times T_n are defined by a kind of Miwa transform ($\eta=\Lambda-\alpha\Lambda^{-1}$):

$$T_n=n^{-1}\,\mathrm{tr}\,\eta^{-n}-\tfrac{1}{2}N\delta_{n2}\quad \text{for } n\geqslant 1 , \qquad T_0=\mathrm{tr}\log\eta^{-1} . \qquad (1.7)$$

Thus the Kontsevich–Penner model may be treated as an intermediate link between 2D gravity described by the Kontsevich model and the random surface technique.

Indeed, as we shall demonstrate, this new model describes in a very natural way the Kontsevich indices for the case of discretized moduli space. If we disregard the moduli space boundary effects (which are relevant in this model), then the coefficients of expansion are just the Kontsevich indices, but since we deal with the closure of the moduli space, the answer is tuned in a way to incorporate the Riemann surfaces which are boundary components under the Deligne and Mumford reduction procedure [19]. On taking the scaling limit (but keeping N finite), we get just the Kontsevich model. On the other hand, using another rescaling we obtain the Penner model describing virtual Euler characteristics of the moduli space via the cell decomposition. Thus, this model provides a bridge between Harer, Zagier and Penner theory [20,21] describing virtual Euler characteristics on moduli space, and the Kontsevich theory giving intersection numbers of stable cohomology classes on the moduli space.

2. The geometric approach to the Kontsevich model

In his original paper [5] Kontsevich proved that

$$\sum_{d_1,...,d_s=0}^{\infty}\langle\tau_{d_1},\tau_{d_2},...,\tau_{d_s}\rangle\prod_{i=1}^{s}(2d_i-1)!!\lambda_i^{-(2d_i+1)}$$

$$=\sum_{\Gamma}\frac{2^{-n_0}}{\#\mathrm{Aut}(\Gamma)}\prod_{\{ij\}}\frac{2}{\lambda_i+\lambda_j} , \qquad (2.1)$$

where the objects in angular brackets on the left-hand side are (rational) numbers describing intersection indices, the sum on the right-hand side runs over all oriented connected trivalent "fat graphs" Γ, s labels boundary components, regardless of the genus, n_0 is the number of vertices of Γ, the product runs over all the edges in the graph and #Aut is the volume of the discrete symmetry group of the graph Γ.

The amazing result of Kontsevich is that the quantity on the right-hand side of (2.1) is equal to a free energy in the following matrix model:

$$e^{F_N(\Lambda)} = \frac{\int dX \exp(-\frac{1}{2}\operatorname{tr}\Lambda X^2 + \frac{1}{6}i\operatorname{tr}X^3)}{\int dX \exp(-\frac{1}{2}\operatorname{tr}\Lambda X^2)}, \tag{2.2}$$

where X is an $N\times N$ hermitian matrix and $\Lambda = \operatorname{diag}(\lambda_1, ..., \lambda_N)$. The distinctive feature of expression (2.1) is that in spite of the fact that each selected diagram has the quantity $(\lambda_i+\lambda_j)$ in the denominator, when taking a sum over all diagrams of the same genus and the same number of boundary components all these quantities cancel with the ones from the nominator.

Feynman rules for the Kontsevich matrix model are the following: as in the usual matrix models, we deal with so-called "fat graphs" or "ribbon graphs" with propagators having two sides, each carrying a corresponding index. The Kontsevich model differs from the standard one-matrix hermitian model since there appear additional variables λ_i associated with index loops in the diagram, the propagator being equal to $2/(\lambda_i+\lambda_j)$, where λ_i and λ_j are variables of the two cycles (perhaps the same cycle) which the two sides of the propagator belong to. Also there are trivalent vertices presenting the cell decomposition of the moduli space.

It is instructive to consider the simplest example of genus zero and three boundary components, which we symbolically label λ_1, λ_2 and λ_3. There are two kinds of diagrams giving a contribution in this order (fig. 1). The contribution to the free energy arising from this sum is

$$\frac{1}{6(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)} + \frac{1}{3}\left\{\frac{1}{4\lambda_1(\lambda_2+\lambda_1)(\lambda_3+\lambda_1)} + (1\to 2, 2\to 3, 3\to 1) + (1\to 3, 3\to 2, 2\to 1)\right\}$$

$$= \frac{2\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3(\lambda_2+\lambda_3) + \lambda_1\lambda_3(\lambda_1+\lambda_3) + \lambda_1\lambda_2(\lambda_1+\lambda_2)}{12\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}$$

$$= \frac{1}{12\lambda_1\lambda_2\lambda_3}. \tag{2.3}$$

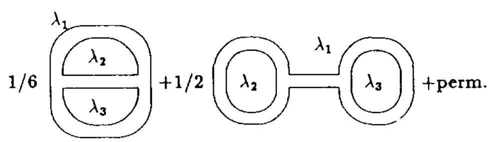

Fig. 1. The $g=0$, $s=3$ contribution to the Kontsevich model.

This example demonstrates the above mentioned cancellations of $(\lambda_i+\lambda_j)$-terms in the denominator.

Now a sketch of Kontsevich's proof is in order. Let us associate with each edge e_i of a fat graph its length $l_i>0$. We consider the orbispace $\mathcal{M}_{g,n}^{\text{comb}}$ of fat graphs with all possible lengths of edges and arbitrary valences of vertices. Two graphs are equivalent if there exists an isomorphism between them. Let us introduce an important object, the space of $(2,0)$-meromorphic differentials $\omega(z)\,\mathrm{d}z^2$ on a Riemann surface with g handles and n punctures; the only poles of $\omega(z)$ are n double poles placed at the punctures with strictly positive quadratic residues $p_i^2>0$ $(i=1, ..., n)$. It is Strebel's theorem [22] which claims that the natural mapping from $\mathcal{M}_{g,n}^{\text{comb}}$ to the moduli space $\mathcal{M}_{g,n}\times\mathbb{R}_+^n$, where $\mathbb{R}_+^n$ is the space of residues, $p_i>0$ being perimeters of cycles, is a homeomorphism. Thus, varying l_j and taking the composition of all graphs we span the whole space $\mathcal{M}_{g,n}\times\mathbb{R}_+^n$.

Each cycle can be interpreted as a boundary component I_i of the Riemann surface since in the Strebel metric it can be presented as a half-infinite cylinder with the puncture placed at infinity. Its boundary consists of a finite number of intervals (edges). We consider a set of line bundles $\mathcal{L}_i$ whose fiber at a point $\Sigma\in\mathcal{M}_{g,n}$ is the cotangent space to the puncture point x_i on the surface Σ. The first Chern class of the line bundle $\mathcal{L}_i$ admits a representation in terms of the lengths of the intervals l_j. The perimeter of the boundary component is $p_i=\sum_{l_\alpha\in I_i} l_\alpha$ and

$$c_1(\mathcal{L}_i)=\sum_{\substack{a,b\in I_i\\ a<b}}\mathrm{d}(l_a/p_i)\wedge\mathrm{d}(l_b/p_i)\,, \tag{2.4}$$

where cyclic ordering is assumed. Following Kontsevich we introduce the two-form Ω:

$$\Omega=\sum_{i=1}^{n}\tfrac{1}{2}p_i^2c_1(\mathcal{L}_i)\,. \tag{2.5}$$

The intersection indices are generated by integrals over the appropriate power $d=3g-3+n$ of the form Ω:

$$\frac{2^d}{d!}\int_{\mathcal{M}_{g,n}}\Omega^d=\frac{1}{d!}\int_{\mathcal{M}_{g,n}}[p_1^2c_1(\mathcal{L}_1)+\cdots+p_n^2c_1(\mathcal{L}_n)]^d$$

$$=\sum_{\Sigma d_i=d}\prod_{i=1}^{n}\frac{p_i^{2d_i}}{d_i!}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\,. \tag{2.6}$$

One important note is in order. It is a theorem by Kontsevich that these integrations extend continuously to the closure of the moduli space $\bar{\mathcal{M}}_{g,n}$ following the procedure by Deligne and Mumford [19], and the proper integration goes over $\bar{\mathcal{M}}_{g,n}\times\mathbb{R}_+^n$. (This means that we deal with a stable cohomological class of curves.)

Taking the Laplace transform over the variables p_i we get

$$\int_0^\infty \mathrm{d}p_i\, \mathrm{e}^{-p_i\lambda_i} p_i^{2d_i} = (2d_i)!\lambda_i^{-2d_i-1} \tag{2.7}$$

for the quantities standing on the right-hand side of (2.6). On the left-hand side we have

$$\int_0^\infty \cdots \int_0^\infty \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n\, \mathrm{e}^{-\Sigma p_i\lambda_i} \int_{\mathcal{M}_{g,n}} \mathrm{e}^{\Omega}, \tag{2.8}$$

and due to cancellations of all p_i^2 multipliers with p_i's in denominators of the form Ω we get

$$\mathrm{e}^{\Omega}\, \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n = c \prod_\alpha \wedge\, \mathrm{d}l_\alpha\,. \tag{2.9}$$

Surprisingly, the constant c depends only on the Euler characteristic of the graph Γ, $c = 2^{-\kappa}$, $\kappa = 2 - 2g = \#\text{vertices} - \#\text{edges} + n$. Thus we have

$$\sum_{\Sigma d_i = d} \prod_{i=1}^{n} \frac{(2d_i-1)!!}{\lambda_i^{2d_i+1}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g$$

$$= \sum_\Gamma \frac{2^{-\kappa}}{\#\mathrm{Aut}\,\Gamma} \int [\mathrm{d}l] \exp\left(\sum_\alpha l_\alpha (\lambda_\alpha^{(1)} + \lambda_\alpha^{(2)}) \right). \tag{2.10}$$

Here $\lambda_\alpha^{(1)}$ and $\lambda_\alpha^{(2)}$ are variables of two cycles divided by the αth edge. Integration over all $\mathrm{d}l_\alpha$ gives us eventually relation (2.1).

3. The Penner–Kontsevich model

Now let us turn to the case of the Penner–Kontsevich model (PK model). Unlike the Kontsevich model, it includes all powers of X^n in the potential since it describes the partition of moduli space into cells of a simplicial complex, the sum running over all simplices with different dimensions. (In the language of the Kontsevich model, the lower the dimension is, the more and more edges of the fat graph are reduced.) Then the virtual Euler characteristic is obtained by weighting the simplices by

$$(-1)^{d_F}/|G_F|\,, \tag{3.1}$$

where $|G_F|$ denotes the order of a stabilizer of the subgroup of the mapping class group, that is, the order of the symmetry group of the corresponding fat graph. It is the Penner model which gives the answer for the sum over F in (3.1) as a free energy for a matrix model [21,23]:

$$\sum_F N^{2-2g}t^{2-2g-n}\frac{(-1)^{d_F}}{|G_F|}=\log\int \mathrm{d}X\,\mathrm{e}^{Nt\,\mathrm{tr}[\log(1-X)+X]}\,, \tag{3.2}$$

where n is the number of punctures on a genus g Riemann surface. Expansion of the free energy of this model in N and t reveals logarithmic corrections, which is a feature of $c=1$ theories.

We find the Feynman rules for the Kontsevich–Penner theory (1.5). First, as in the standard Penner model, we have vertices of all orders in X. Due to rotational symmetry, the factor $1/n$ standing with each X^n cancels, and only the symmetrical factor (3.1) survives. Moreover, there is a factor $(-\alpha)$ standing with each vertex. As in the Kontsevich model, there are variables λ_i associated with each cycle. But the form of the propagator changes, instead of $2/(\lambda_i+\lambda_j)$ we have $1/(\lambda_i\lambda_j+\alpha)$.

Let us consider the same case ($g=0$, $n=3$) as for Kontsevich model. One additional diagram resulting from the vertex X^4 arises and gives a contribution with opposite sign (fig. 2). This contribution is (symmetrized over λ_1, λ_2 and λ_3):

$$\begin{aligned}&-\frac{1}{3}\left\{\frac{\alpha}{2(\lambda_1\lambda_2+\alpha)(\lambda_1\lambda_3+\alpha)}+\text{perm.}\right\}\\&+\frac{\alpha^2}{6(\lambda_1\lambda_2+\alpha)(\lambda_1\lambda_3+\alpha)(\lambda_2\lambda_3+\alpha)}\\&+\frac{1}{3}\left\{\frac{\alpha^2}{2(\lambda_1^2+\alpha)(\lambda_1\lambda_2+\alpha)(\lambda_1\lambda_3+\alpha)}+\text{perm.}\right\}.\end{aligned} \tag{3.3}$$

Again collecting all terms we get

$$\begin{aligned}&\frac{\alpha}{6\prod_{i<j}(\lambda_i\lambda_j+\alpha)}\left\{-\sum_{i<j}\lambda_i\lambda_j-2\alpha\right.\\&\left.+\alpha\left(\frac{\lambda_2\lambda_3+\alpha}{\lambda_1^2+\alpha}+\frac{\lambda_1\lambda_2+\alpha}{\lambda_3^2+\alpha}+\frac{\lambda_1\lambda_3+\alpha}{\lambda_2^2+\alpha}\right)\right\},\end{aligned} \tag{3.4}$$

and after a little algebra we obtain the answer:

$$F_{0,3}=\alpha\frac{-\lambda_1\lambda_2-\lambda_1\lambda_3-\lambda_2\lambda_3+\alpha}{3(\lambda_1^2+\alpha)(\lambda_2^2+\alpha)(\lambda_3^2+\alpha)}\,. \tag{3.5}$$

We see that here, just as in the standard Kontsevich model, cancellation of inter-

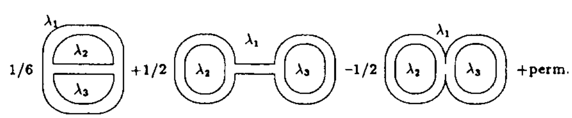

Fig. 2. $g=0$, $s=3$ contribution to the Penner–Kontsevich model.

twining terms in the denominator occurs leading to factorization of the answer in $1/(\lambda_i^2+\alpha)$ terms. It should reveal an underlying geometric structure of the model under consideration, where quantities like (2.1) are expected to arise.

4. Relation between the KP model and discretized moduli space

Now we turn to the description of the underlying differential-geometric structure of the Kontsevich–Penner model. Let us consider the case of a discretized moduli space. Its description can be done most properly in terms of lengths of edges constituting boundary components of the surface. We assume all these lengths to be integers (probably zero) scaled by some factor ϵ. Thus, the perimeters p_i now belong to $\epsilon\cdot\mathbb{Z}_+$. Then we can present the first Chern class of the line bundle $\mathscr{L}_i$ in the same form as above,

$$c_1(\mathscr{L}_i)=\sum_{\substack{a,b\in I_i\\ a<b}} \mathrm{d}(n_a/p_i)\wedge\mathrm{d}(n_b/p_i)\ , \tag{4.1}$$

where $\mathrm{d}n_a$ are symbols satisfying standard relations, $\mathrm{d}n_a$ lie in the cotangent space to the continuous moduli space taken at a point $(n_1, ..., n_{6g-6+3n})$ which also belongs to the discretized moduli space. The action of the external derivative d follows the same rules as in the continuous case and the integration is replaced by a discrete half-infinite sum. The two-form Ω is defined by the same formula (2.5), thus the "volume formula" (2.9) is also preserved. Subtleties appear when one has to integrate over discrete moduli space $\mathscr{M}_{g,n}^{\mathrm{disc}}$. First, there are points in $\mathscr{M}_{g,n}^{\mathrm{disc}}$ which do not lie on the boundary of the moduli space but at which some of the lengths n_a are equal to zero. These points correspond to graphs containing vertices of order greater than three. In the continuum limit we did not take into account such graphs since they correspond to subdomains of lower dimensions in the interior of the moduli space, and because the integration measure is continuous we may neglect them. Now the situation has changed and we should consider these diagrams as well.

Second, now we should explicitly take into account curves which are reduced by the Deligne–Mumford procedure [19]. Then the "boundary" of the moduli space is given by a stratification procedure. Fortunately, we can present an explicit integration over the boundary $\partial\mathscr{M}_{g,n}^{\mathrm{disc}}$ of the discrete moduli space because it expands into a sum over strata which are direct products of connected components of lower genus and a number of additionally inserted punctures corresponding to reduced handles of the surface. Doing all possible reductions we span the whole closure of $\mathscr{M}_{g,n}^{\mathrm{disc}}$:

$$\frac{2^d}{d!}\int_{\mathscr{M}_{g,n}^{\mathrm{disc}}}\Omega^d=\frac{1}{d!}\int_{\bar{\mathscr{M}}_{g,n}^{\mathrm{disc}}}\left(\sum_{i=1}^{n}p_i^2c_1(\mathscr{L}_i)\right)^d-\frac{1}{d!}\int_{\partial\bar{\mathscr{M}}_{g,n}^{\mathrm{disc}}}\left(\sum_{i=1}^{n}p_i^2c_1(\mathscr{L}_i)\right)^{d'} . \tag{4.2}$$

Here $d' < d$ depends on the power of the reduction.

A point $\Sigma \in \partial \mathcal{M}_{g,n}^{\text{disc}}$ is a union of s $(1 \leqslant s \leqslant n+2g-2)$ connected components Σ_{g_j,n_j,k_j} $(j=1, \ldots, s)$. Each surface Σ_{g_j,n_j,k_j} has genus g_j, $\sum_{j=1}^{s} g_j \leqslant g$, n_j original punctures $(\sum_{j=1}^{n} n_j = n)$ and k_j additional punctures $\mathcal{P}_l^{(j)}$ arising from the reduction procedure. The linear bundles $\mathcal{L}_i$ are associated with n_j points of this surface but not with the new $\mathcal{P}_l^{(j)}$. Explicitly the boundary $\partial \mathcal{M}_{g,n}^{\text{disc}}$ can be presented as a finite set of disconnected pieces; each of them is in its turn the direct product of lower dimensional closed moduli spaces weighted with $(-1)^{r_s}$, $r_s = \frac{1}{2}\sum_j k_j$ being the reduction power:

$$\bigotimes_{j=1}^{s} \bar{\mathcal{M}}_{g_j,n_j+k_j}^{\text{disc}}(-1)^{r_s} . \tag{4.3}$$

Integration over $\bar{\mathcal{M}}_{g_j,n_j+k_j}^{\text{disc}}(-1)^{r_s}$ expands into a product of integrals over connected closed components, which are given by known continuous Kontsevich indices. So we conjecture the answer for intersection indices on the discretized moduli space:

$$\frac{2^d}{d!} \int_{\mathcal{M}_{g,n}^{\text{disc}}} \Omega^d = \sum_{\sum d_i = d} \prod_{i=1}^{n} \frac{p_i^{2d_i}}{d_i!} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g + \sum_{\text{reductions}} (-1)^{r_s} \frac{(d-r_s)!}{d!}$$
$$\times \prod_{j=1}^{s} \left(\sum_{\sum d_a = d_j} \prod_{a=1}^{n_j} \frac{p_a^{2d_a}}{d_a!} \langle \tau_{d_1} \cdots \tau_{d_{n_j}} \tau_0^{(1)} \cdots \tau_0^{(k_j)} \rangle_{g_j} \right), \tag{4.4}$$

where $\langle \cdots \rangle_{g_j}$ are the corresponding Kontsevich indices, $d_j = 3g_j - 3 + n_j + k_j$ and k_j insertions of τ_0 correspond to additional punctures.

The next step is to Laplace transform of the variables p_i. Now we should sum over $p_i = \epsilon, 2\epsilon, 3\epsilon, \ldots$ simultaneously taking into account that $\sum_{i=1}^{n} p_i \in 2\epsilon \mathbb{Z}_+$ (every edge n_a is counted twice). A procedure is the following. We perform the transform over the variables λ_j taking a sum over all $p_j \in \epsilon \mathbb{Z}_+$ weighted with $(\mathrm{i})^j$ and release the real part of the expression obtained after substituting $\mathrm{e}^{\epsilon\lambda_j} \to \mathrm{i}\mathrm{e}^{\epsilon\lambda_j}$. On the right-hand side we have (explicitly reconstructing the ϵ-dependence)

$$\operatorname{Re} \prod_{j=1}^{n} \left(\sum_{p_j=0}^{\infty} \mathrm{e}^{-\epsilon\lambda_j p_j} (\mathrm{i})^{p_j} \epsilon^{2d_j} p_j^{2d_j} \right) = \prod_{j=1}^{n} \left(\frac{\partial}{\partial \lambda_j} \right)^{2d_j} \operatorname{Re} \frac{(-1)^n}{\prod_{j=1}^{n} (1 + \mathrm{i}\mathrm{e}^{\epsilon\lambda_j})} . \tag{4.5}$$

Taking as an example the case of $\mathcal{M}_{0,3}$ we immediately get expression (3.5).

On the right-hand side we get

$$\sum_{\{p_i\} \in \mathcal{M}_{g,n}^{\text{disc}}} \mathrm{e}^{-\epsilon\lambda_j p_j} \epsilon^n \mathrm{e}^{\Omega} \, \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n$$
$$= \int_{\mathcal{M}_{g,n}^{\text{disc}}} \mathrm{e}^{-\epsilon\lambda_j p_j} 2^{2g-2} \, \mathrm{d}n_1 \wedge \cdots \wedge \mathrm{d}n_{6g-6+3n} \, \epsilon^{6g-6+3n} . \tag{4.6}$$

The last term possesses an explicit representation as a sum over all possible graphs with fixed genus g and number of cycles n and *arbitrary* valences of the vertices:

$$\sum_{\Gamma} \frac{1}{\#\mathrm{Aut}(\Gamma)} 2^{\#\mathrm{edges}-\#\mathrm{vert.}+n} \epsilon^{6g-6+3n} \prod_{s} \sum_{n_s=1}^{\infty} \mathrm{e}^{-\epsilon n_s(\lambda_s^{(1)}+\lambda_s^{(2)})}$$
$$= 2^n \sum_{\Gamma} \frac{1}{\#\mathrm{Aut}(\Gamma)} 2^{-\#\mathrm{vert.}} \epsilon^{6g-6+3n} \prod_{\{ij\}} \frac{2}{\mathrm{e}^{\epsilon(\lambda_i+\lambda_j)}-1}$$
$$\equiv w_g(\lambda_1, \ldots, \lambda_n) \,. \tag{4.7}$$

This last expression is in fact the free energy term for the Kontsevich–Penner matrix model with fixed g and n of the form

$$\mathrm{e}^{F_N(\Lambda)} = \frac{\int \mathrm{D}X \exp N\alpha\, \mathrm{tr}\{-\frac{1}{4}\Lambda X \Lambda X - \frac{1}{2}[\log(1-X)+X]\}}{\int \mathrm{D}X \exp N\alpha\, \mathrm{tr}(-\frac{1}{4}\Lambda X \Lambda X + \frac{1}{4}X^2)} \,, \tag{4.8}$$

where $\alpha = 1/\epsilon^3$ and $\Lambda = \mathrm{diag}(\mathrm{e}^{\epsilon\lambda_1}, \mathrm{e}^{\epsilon\lambda_2}, \ldots, \mathrm{e}^{\epsilon\lambda_N})$. $F_N(\Lambda)$ has an expansion which looks just like (1.3):

$$F_N(\Lambda) = \sum_{\substack{g=0\\n=1}}^{\infty} (N\alpha)^{2-2g} \alpha^{-n} N^{-n}\, \mathrm{tr}\, w_g(\lambda_1, \ldots, \lambda_n) \,. \tag{4.9}$$

In the continuum limit $\epsilon \to 0$ we immediately get expression (2.2), i.e. the Kontsevich matrix model.

Thus we have demonstrated the relation between geometric characteristics of the discretized moduli space and related matrix models. We have shown how the Kontsevich indices can be naturally embedded into the standard one-matrix model via the Penner–Kontsevich model.

In conclusion we should note some problems and perspectives of the proposed model. First, we did not yet prove rigorously relation (4.4), and this proof is necessary for the completeness of the theory. Also it is interesting to take the sum over reductions more explicitly (in the combinatorial sense). A second interesting problem to solve is to find the description of just the Penner model in the case of the discretized moduli space. Formula (3.2) should be modified since in this case the volume of the stabilizer (symmetry group) taken at the (infinite) boundary point might not be infinite but rather proportional to some positive power of ϵ, the discretization parameter. Also it is interesting to develop this approach for the case of GKM, where most subtle geometric invariants are considered.

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